Equidimensional morphisms onto splinters are pure

4/24

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1. Motivation:
   Hochster-Roberts theorem,
   Boutot-type theorems
2. Main Result Ⓘ
3. Weak Boutot-type theorems
   (Main Result (II))

§1 Motivation: Hochster-Roberts & Boutot-type theorems

Thm [Hochster-Roberts '74]

k = field

S = reg ring f.t. $/k$

$\circlearrowright$ k-rationally

G = lin red gp

$\Rightarrow S^G$ CM

Immediate conseq's

char$(k)=0$ The following are arithmetically CM:

- generic det'l var's [Hochster-Eagon '71]

- symm det'l var's [Kutz '74]
- Grassmannian / Schubert var's [Hochster, Laksov, Musili ~72-74]

<u>all char's</u>

- normal affine toric var's are CM [Hochster '72]

<u>Even more generally</u>

<u>Thm</u>

$$\begin{tikzcd} S = \text{reg. ring} \\ R \arrow[u, hook, "\text{pure}"] \end{tikzcd}$$

$\Longrightarrow$ R ① CM

② has rat'l sing's

Def [Cohn '59; Olivier '70]

$\varphi : R \to S$ ring map

φ is *pure* if

$$M \otimes_R R \xrightarrow{id \otimes \varphi} M \otimes_R S$$

is inj. $\forall$ R-mod's M.

Ex · $S =$ f.t. $| k$ field

$\circlearrowright$ k-rat

$G =$ lin red

Reynolds operator

· split maps
$$\begin{tikzcd} R \arrow[r] & S \arrow[l, dashed, bend left] \end{tikzcd}$$

left inv in Mod(R)

· faithfully flat

Rem [Olivier '71]

$\varphi: R \to S$ pure

$\Longleftrightarrow$ eff. descent mor. for mod's

	①	②
char. $p>0$	HR '74	Smith '97
$R \supseteq \mathbb{Q}$	Kempf '79 eft|k Hochster–Huneke '95	Boutot '87 eft|k Schoutens '08
mixed char.	Heitmann–Ma '18	

More generally:

Q $R \hookrightarrow S$ pure. What prop's of S descend to R?

Ex's ① Noeth [Olivier '73]

② normality [Hochster-Roberts '74]

③ splinter [Datta-Tucker '23]

Def [Ma '88]

R = Noeth ring

R **splinter** if, $\forall$ mod-fin $R \xrightarrow{\varphi} S$, s.t. $\operatorname{Spec}(S) \twoheadrightarrow \operatorname{Spec}(R)$ surj,

φ is pure

Thm [Boutot '87; M- '25]

R ↪ S ~~~~eff'l/k~~~~ pure map of ~~~~general~~~~ Noeth $\mathbb{Q}$-alg's

S has rat'l sing's

⟹ R has rat'l sing's

= "Boutot-type thm"

Ex's Noeth $\mathbb{Q}$-alg's

① klt type sing's (exc. + wẍ)

[Braun-Greb-Langlois-Moraga for G red;

Zhuang-Lyu '24]

② Du Bois sing's

[Godfrey-M]

③ Dense F-pure / F-inj type [Yamaguchi '25]

Noeth $\mathbb{F}_p$-alg's

① F-pure [HR'76]

② F-regular [HH'90; Hashimoto '10]

③ FALSE for F-rational & F-injective [Watanabe '97] [Nguyen '12]

Main Q's

Ⓘ $\exists$ other interesting ex's of pure maps?

Ⓘⓘ $\exists$ Weak Boutot-type thm's where "pure" replaced by "pure + something"?

§2 Main Result Ⓘ

Thm [M-] Y = loc. Noeth. sch.

$\mathcal{O}_{Y,y}$ splinters $\forall y \in Y$

$\iff$ every locally equidim'l surj $X \xrightarrow{f} Y$ is strongly pure $\left(\begin{array}{l} \forall x \in X, \\ \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x} \\ \text{is pure} \end{array} \right)$

$\Leftarrow$: Easy: Finite $\Rightarrow$ loc. equidim'l

$\Rightarrow$: Analogue of "miracle flatness"

($X \xrightarrow{f} Y$ fib. of var's, f loc. equidi'l
Y reg, X CM $\Rightarrow$ f flat)

Generalizes

- [Hochster '73; Ma'88]
  Noeth. $\mathbb{Q}$-alg's
  normal $\Longleftrightarrow$ splinter
- [Chakraborty–Gurjar–Miyanishi '16; Lyu '24]
  Noeth. normal $\mathbb{Q}$-alg's
  $\Longrightarrow$ every fas in the Thm
  is partially pure

Key pt "splinter" can replace "normal" outside equichar 0

What does "locally equidim" mean?

Thm [Chevalley '56; EGA IV_3]

$f: X \to Y$ dom't loc. f.t.

X, Y irred

$\ni$

η gen pt

$$\underbrace{\dim_x}\left(f^{-1}(f(x))\right) \geq \dim\left(f^{-1}(\eta)\right)$$

$\forall x \in X$

max dim of open $\mathcal{U} \ni x$

(when f.t., $\Leftrightarrow$ max dim of irr cpt $\ni x$)

Roughly speaking: locally equidim if "=" always holds

New Ex's of pure maps

$f : X \to Y$ family of var's all same dim, Y splinter $\Rightarrow f$ pure

Thm [Gross '94] $/\mathbb{C}$

X = proj. var., $K_{X^m} = 0$ ← 3fold

$\downarrow$ ell. fib.

Y = proj. var.

is birat'l to $\bar{X}$ = min'l

equidim $\downarrow$ ell fib

$\bar{Y}$ = A_n sing's

<u>Ex</u> Gross constructs exs where

$\overline{Y}$ <u>has</u> A_1 sing's.

Such an example cannot be flat.

Prop [EGA IV_3] $f : X \to Y$ mor. of schemes, loc. f.t., $x \in X$ pt

TFAE:

ⓐ $\exists\, e \in \mathbb{N}$, open nbhd $\mathcal{U} \ni x$, s.t. every irr cpt of $\mathcal{U}$ dom's an irr comp of Y and, $\forall x' \in \mathcal{U}$,

$$\mathcal{U} \cap f^{-1}(f(x'))$$

equidim'l of dim e

ⓑ $\exists\, e \in \mathbb{N}$, open nbhd $\mathcal{U} \ni x$ and factorization

$$\begin{tikzcd} \mathcal{U} \arrow[r, "g"] \arrow[dr, "f|_{\mathcal{U}}"'] & \mathbb{A}^e_Y \arrow[d, "p"] \\ & Y \end{tikzcd}$$

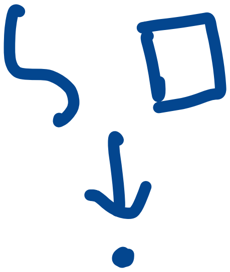
g quasi-fin

s.t. every irrcpt of $\mathcal{U}$ dom's an irrcompt of $\mathbb{A}^e_Y$.

Def [EGA IV_3; EGA IV_4, Errata]

- f **equidimensional at x** if ⓐ or ⓑ holds.
- f **locally equidimensional** if equidim'l at every $x \in X$
- f **equidim'l** if loc. equidim'l and $f^{-1}[f(x))$ equidim'l $\forall x \in X$

Ex



loc equidim

not equidim

Prop [M-]

ⓑ can be strengthened so that, if x_0 is a gen pt of $f^{-1}(f(x))$ of an irr cpt $\ni x$, then we can find a factorization as in ⓑ

$$x_0 \mapsto \text{gen pt of } p^{-1}(f(x)) \cong \mathbb{A}^e_{k(f(x))}$$

Key Idea Apply Noether normalization to $f^{-1}(f(x))$:

Thm [Noether 1926; Bourbaki '64]

A = f.g. / k field

$\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \cdots \subseteq \mathfrak{a}_p \subsetneq A$

ideals in A s.t. $p \geq 1$

$\Rightarrow \exists\ x_1, \ldots, x_n$ alg indep in A s.t.

① $k[\underline{x}] \hookrightarrow A$ integral

② $\exists$ inc. seq. $(h(j))_{j=1}^{p}$ s.t.

$\sigma_j \cap k[\underline{x}] = (x_1, \ldots, x_{h(j)})$

$\forall j$

Pf of Thm $\Rightarrow$

Reduce to X, Y ired. t aff.

$Y = \mathrm{Spec}(R)$, $(R, \mathfrak{m})$ local

Prop $\Rightarrow$

$$\begin{tikzcd}
S & & R[\underline{x}] \arrow[ll, "\text{qfin}"'] \\
\rotatebox[origin=c]{90}{$\subsetneq$} & & \\
\mathfrak{q} & R \arrow[uul] \arrow[uur] &
\end{tikzcd}$$

gen pt of irr cpt $\ni x$

s.t. $\mathfrak{q} \cap (R/\mathfrak{m})[\underline{x}] = 0$

localize

$$\begin{tikzcd}
S_{\mathfrak{q}'} & & (R[\underline{x}])^{h}_{\mathfrak{m}R[\underline{x}]} \arrow[ll, "\text{fin}"', "\text{b/c Henselian}"] \\
S_{\mathfrak{q}} \arrow[u] & & (R[\underline{x}])_{\mathfrak{m}R[\underline{x}]} \arrow[ll, "\text{qfin}"] \arrow[u] \\
& R \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

[Lyu '25] $(R[\underline{x}])^{h}_{\mathfrak{m}R[\underline{x}]}$ splinter

$\Rightarrow$ top map pure

$\Rightarrow$ $R \to S_{\mathfrak{q}}'$ pure

$\Rightarrow$ $R \to S_{\mathfrak{q}}$ pure $\square$

§3 Weak Boutot-type thm's

(II) ∃ Weak Boutot-type thm's where "pure" is replaced by "pure + sthg"?

Yes, e.g.:

[Nakayama '10] $\mathbb{Q}$-factorial
surj, loc. equidim'l mor. btw.
integral, normal, Noeth. sep. sch's

[M '21; Datta-M '24]
F-injective pure + quasi-fin
or strongly pure

Thm [M-] $R \hookrightarrow S$ pure map Noeth prime char. $p>0$

Spec$(S) \to$ Spec(R) loc. equidim'l

S loc. F-rat'l + R univ. cat

$\Rightarrow R$ loc. F-rat'l

Idea · Reduce to R, S local

NTS $\forall$ param. $I \subseteq R$, $I^* = I$

· Show IS param. ideal

(use loc. equid'l + R univ. cat.

$\Rightarrow IS$ can be completed to a sop)

· $(IS)^* = IS$

$\Rightarrow I^* \subseteq (IS)^* \cap R = IS \cap R$

$= I$ (purity)